State Space Design: Pole Placement for State Feedback & Observers

MEM 355 Performance Enhancement of Dynamical Systems

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Outline

State space techniques emerged around 1960. They are direct and exploit the efficient computations of linear algebra.

- State feedback & pole placement
- Observers
- Design via separation principle
- Design examples

State Feedback Pole Placement

Given a linear system:

$$\dot{x} = Ax + Bu$$

find a state feedback control:

$$u = Kx$$

such that the closed loop system:

$$\dot{x} = Ax + BKx = \left(A + BK\right)x$$

has a specified (self-conjugate) set of poles $\left\{p_1, p_2, \ldots, p_n\right\}$.

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Pole Placement Sol'n: SISO Case

- Convert $\dot{x} = Ax + bu$ to controller form (phase variable form) using $x = Tz$:

$$\dot{z} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} z + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

controller form

- Set $u = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} z$ and obtain closed loop: $\dot{z} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ k_1 - a_0 & k_2 - a_1 & \cdots & k_n - a_{n-1} \end{bmatrix} z$

- Expand desired closed loop characteristic polynomial and compare coefficients, and solve for $k_1, \ldots, k_n$:

$$\phi_{cl}(\lambda) = (\lambda - p_1)(\lambda - p_2)\cdots(\lambda - p_n) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0 \Rightarrow \alpha_0 = a_0 - k_1, \alpha_1 = a_1 - k_2, \ldots, \alpha_{n-1} = a_{n-1} - k_n$$

- Convert back to x-coordinates: $Kz = KT^{-1}x \Rightarrow u = \left(KT^{-1}\right)x$

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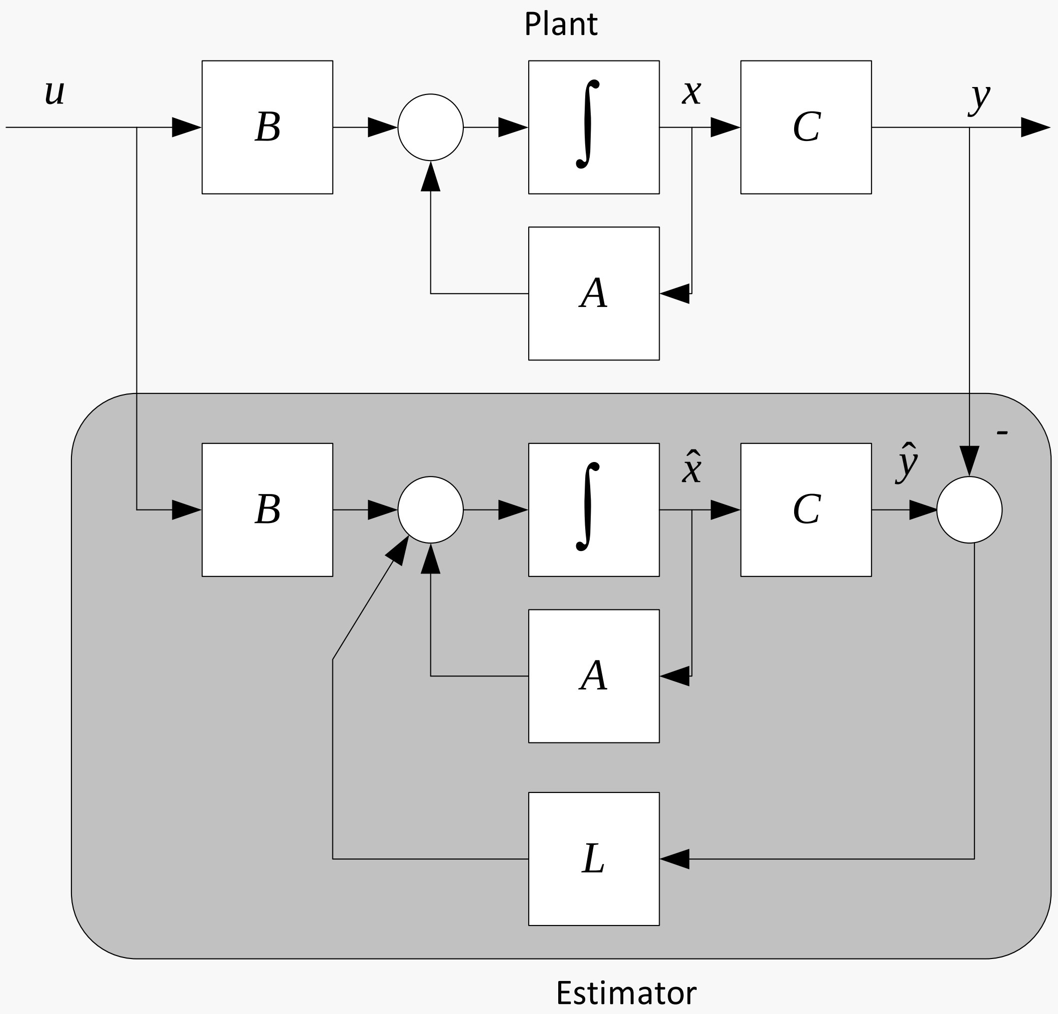
Pole Place Design: The Easy Way

PLACE Pole placement technique

K = PLACE(A,B,P) computes a state-feedback matrix K such that the eigenvalues of A-B*K are those specified in vector P. No eigenvalue should have a multiplicity greater than the number of inputs.

Warning!! Notice the sign difference.

Full-State Estimator



Estimator Error Dynamics

$$\dot{x} = Ax + Bu,\ y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(\hat{y} - y),\ \hat{y} = C\hat{x}$$

$$e := x - \hat{x} \Rightarrow \dot{e} = Ae + LCe \Rightarrow \boxed{\dot{e} = (A + LC)e}$$

One approach is to select L so as to place the poles of $(A + LC)$. Notice that the following two pole placement problems are equivalent:

$(A + BK), (A, B)$ controllable

$(A^T + C^T L^T), (A, C)$ observable

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Closed Loop Dynamics

$$\dot{x} = Ax + Bu$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L\left(C\hat{x} - Cx\right)$$

$$u = K\hat{x}$$

$$\dot{x} = Ax + BK\hat{x} = \left(A + BK\right)x - BKe$$

$$\dot{e} = Ae + LCe = \left(A + LC\right)e$$

$$\boxed{\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A+BK & -BK \\ 0 & A+LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}} \Rightarrow$$

closed loop poles

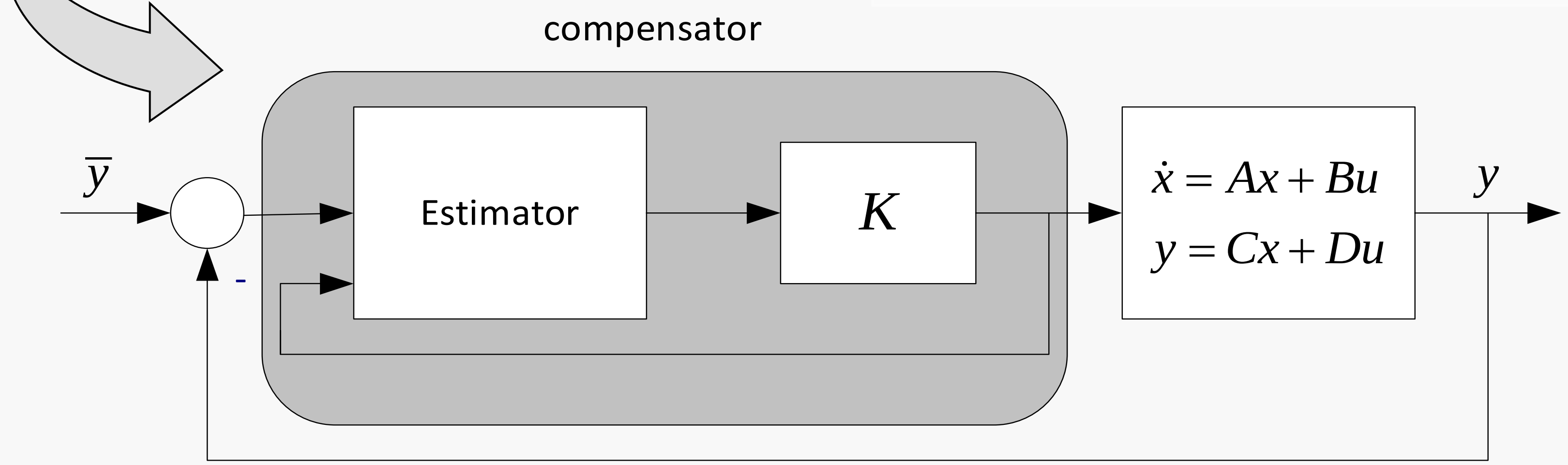
$$\lambda\left(A + BK\right) + \lambda\left(A + LC\right)$$

Compensator

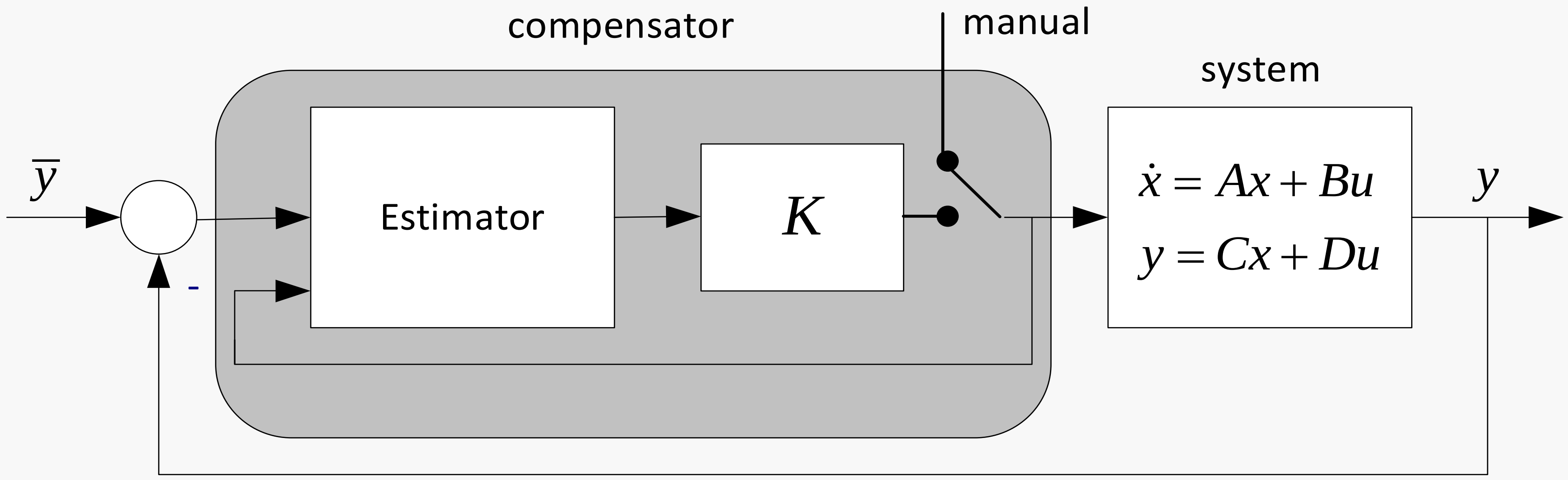
$$\dot{\hat{x}} = A\hat{x} + Bu + L(\hat{y} - y) \Rightarrow (A + BK + LC)\hat{x} - Ly$$

$$u = K\hat{x}$$

$$G_c(s) = K\left[sI - (A + BK + LC)\right]^{-1} L$$



Implementation



- design a state feedback controller $u = Kx$
- design a state estimator/observer to produce $\hat{x}$
- implement the control $K\hat{x}$

Examples

- In the following examples we will
  - Examine open loop modes
  - Design controller using pole placement
  - Compute equivalent compensator
  - Perform root locus analysis of feedback loop

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Example: F-16

landing approach longitudinal dynamics

$$\begin{bmatrix} \dot{u} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.507 & -3.861 & 0 & -32.17 \\ -0.00117 & -0.5164 & 1 & 0 \\ -0.000129 & 1.4168 & -0.4932 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ -0.0717 \\ -1.645 \\ 0 \end{bmatrix} \delta_E$$

Open loop modes

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ \alpha \\ q \\ \theta \end{bmatrix}$$

$$\text{phugoid: } \lambda = -0.0438167 \pm j0.206461 \quad h = \begin{bmatrix} 0.999978 \\ 0.000484 \\ 0.001343 \\ -0.000272 \end{bmatrix} \pm j \begin{bmatrix} 0 \\ 0.0002676 \\ 0.0002264 \\ -0.0064497 \end{bmatrix}$$

$$\text{short period: } \lambda = -1.7036, 0.730937 \quad h = \begin{bmatrix} -0.994287 \\ -0.063373 \\ 0.074073 \\ -0.043481 \end{bmatrix}, \begin{bmatrix} 0.999508 \\ -0.014171 \\ -0.016507 \\ -0.022584 \end{bmatrix}$$

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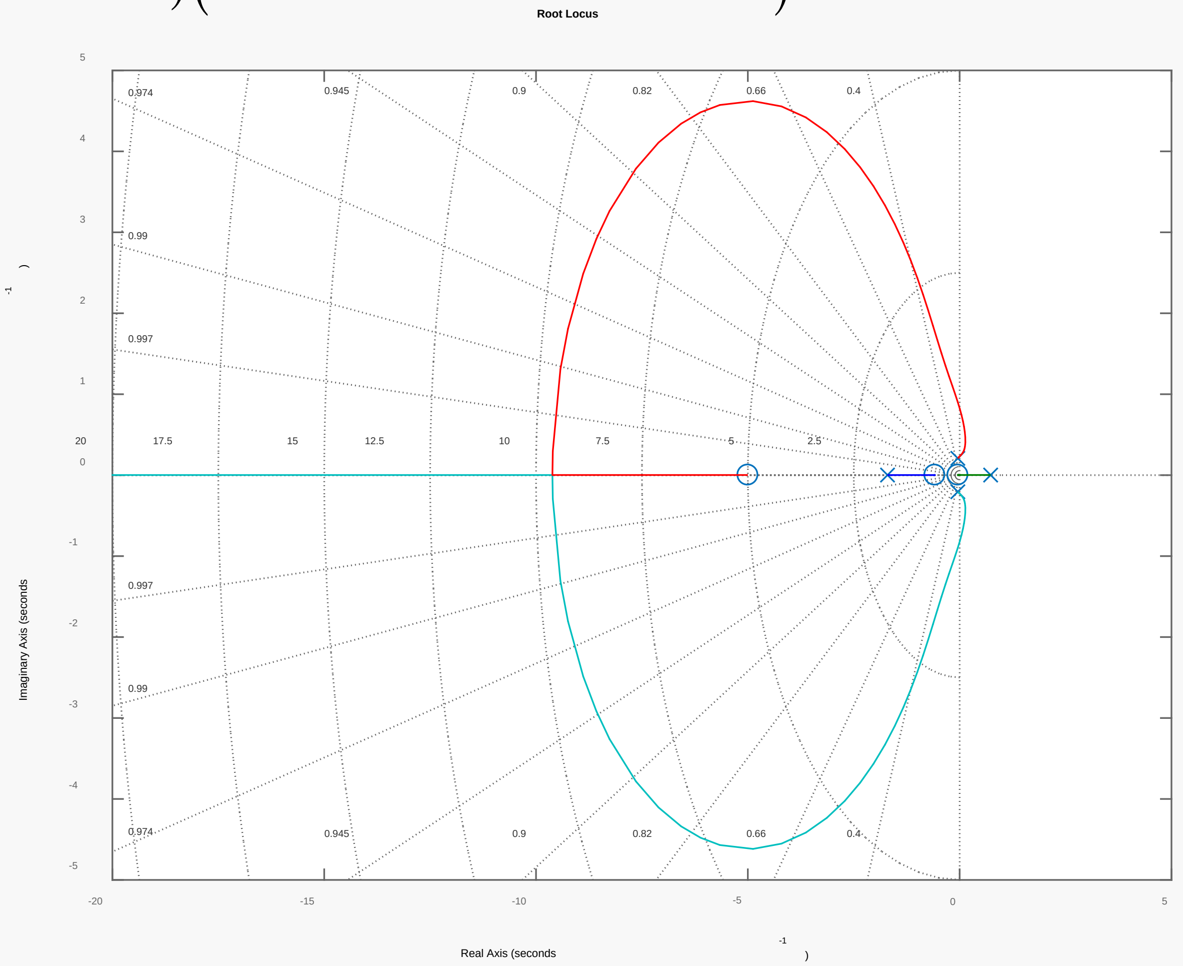
F-16: PI Control

$$G_p(s) = 1.645\frac{s(s+0.0423101)(s+0.586543)}{(s-0.730937)(s+1.7036)(s^2+0.0876334s+0.044546)}$$

$$G_c(s) = \frac{s+5}{s}$$

Design via root locus



Example: F-16 state feedback

Desired poles -

short period: $\lambda_{1,2} = -1.25 \pm j2.16506 \quad (\omega = 2.5, \rho = 0.5)$

phugoid: $\lambda_{3,4} = -0.01 \pm j0.0994987 \quad (\omega = 0.1, \rho = 0.1)$

$$K = \begin{bmatrix} 0.004076 & 3.87578 & 0.718424 & 0.095189 \end{bmatrix}$$

Design via pole placement – requires an observer

Example: F-16 Rynaski “robust observer”

"place observer poles at LHP plant zeros, remainder are placed arbitrarily"

$$\lambda = 0, -0.04231, -0.5865, -1$$

$$L^T = \begin{bmatrix} 0.168343 & -1.02106 & -0.56851 & -1 \end{bmatrix}$$

$$G_p(s) = 1.645 \frac{s(s+0.0423101)(s+0.586543)}{(s-0.730937)(s+1.7036)(s^2+0.0876334s+0.044546)}$$

$$G_c(s) = 4.46035 \frac{(s+1.234)(s^2+0.02967s+0.02198)}{s(s+0.04231)(s+0.5866)(s+2.46)}$$

Equivalent compensator

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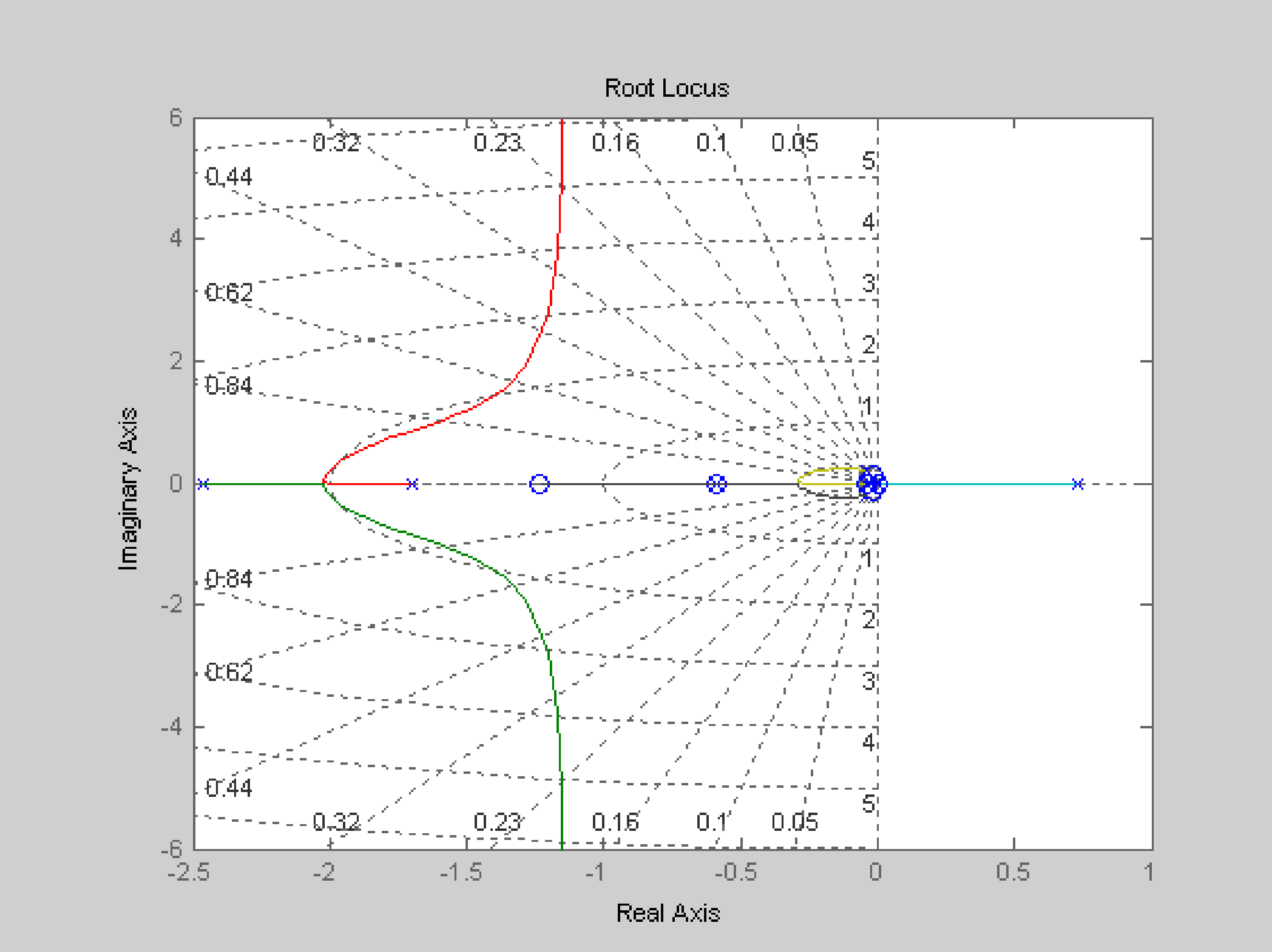
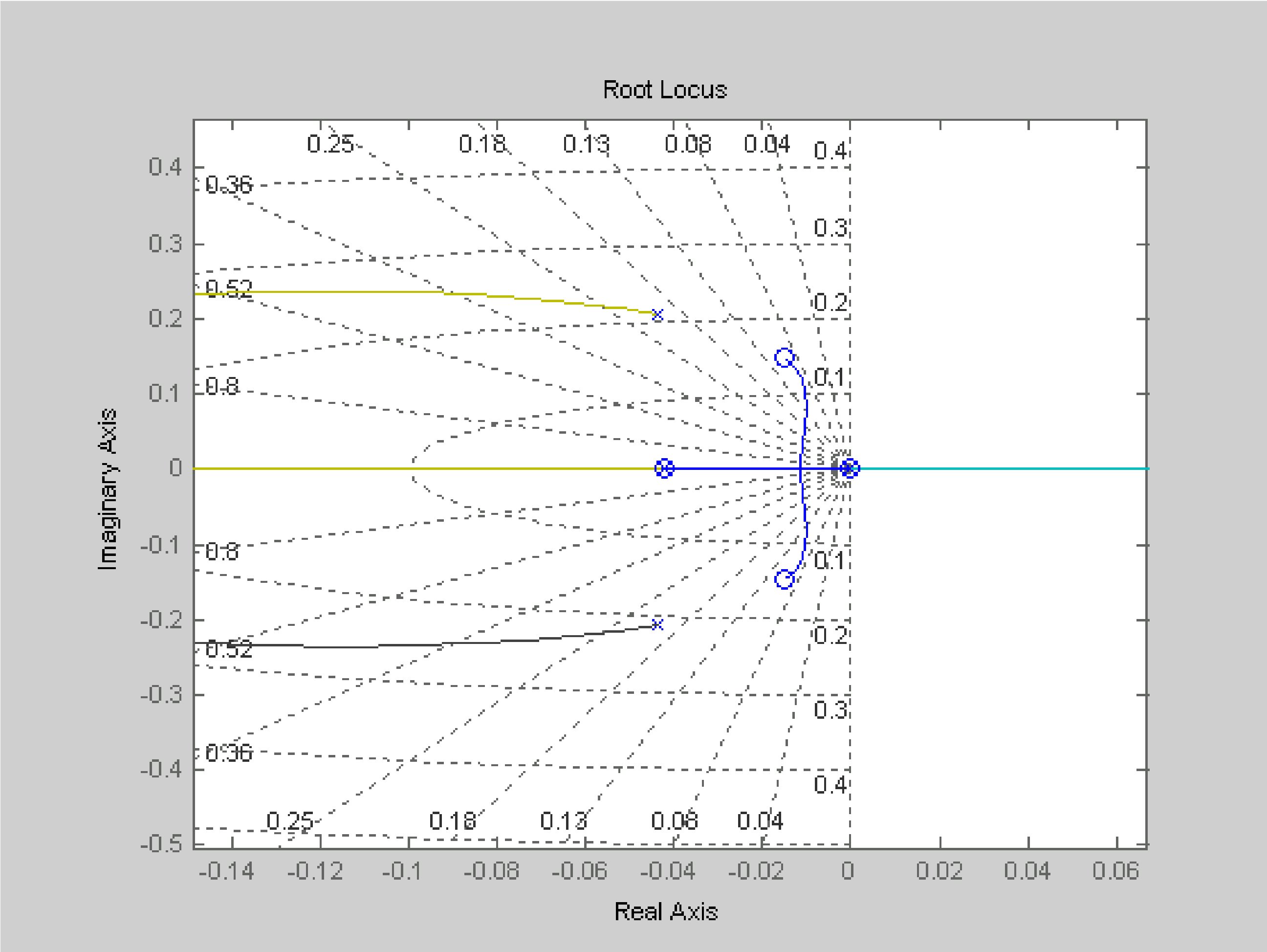
Root Locus

$$G_p(s) = 1.645\frac{s(s+0.0423101)(s+0.586543)}{(s-0.730937)(s+1.7036)(s^2+0.0876334s+0.044546)}$$

$$G_c(s) = 4.46035\frac{(s+1.234)(s^2+0.02967s+0.02198)}{s(s+0.04231)(s+0.5866)(s+2.46)}$$



Root Locus 2



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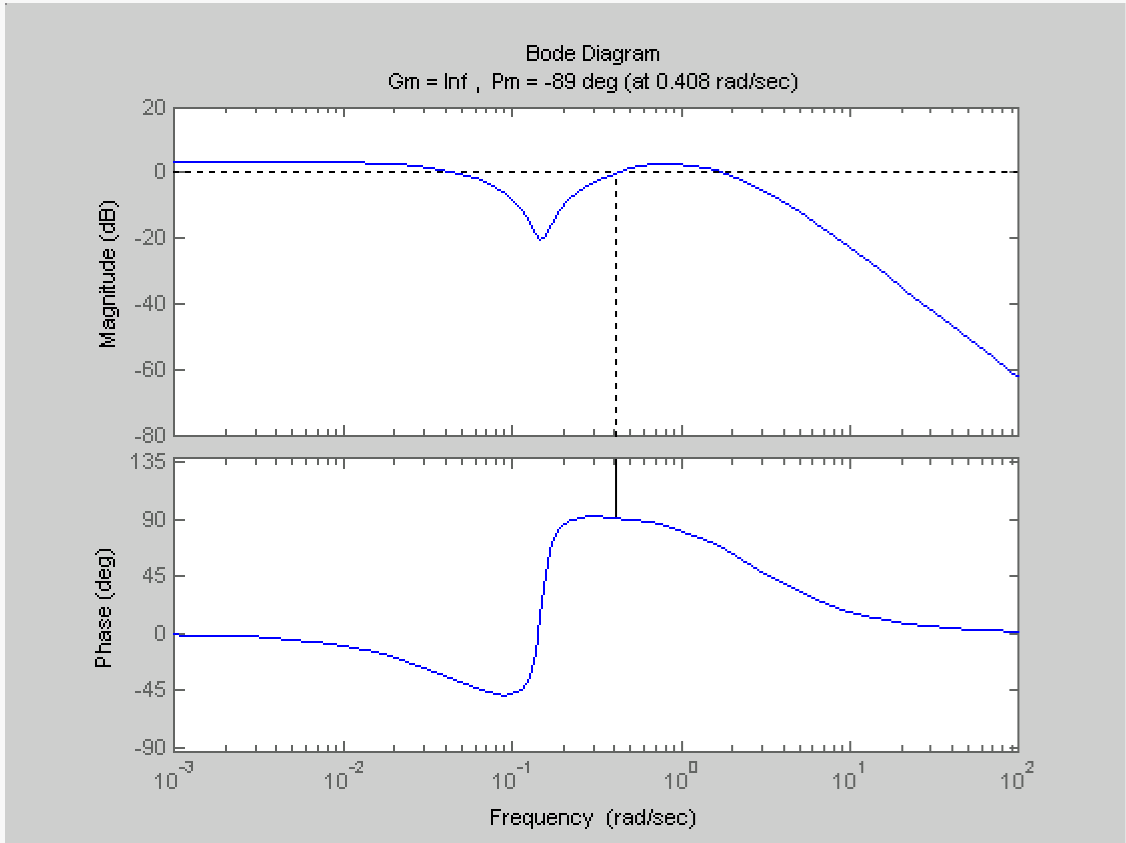
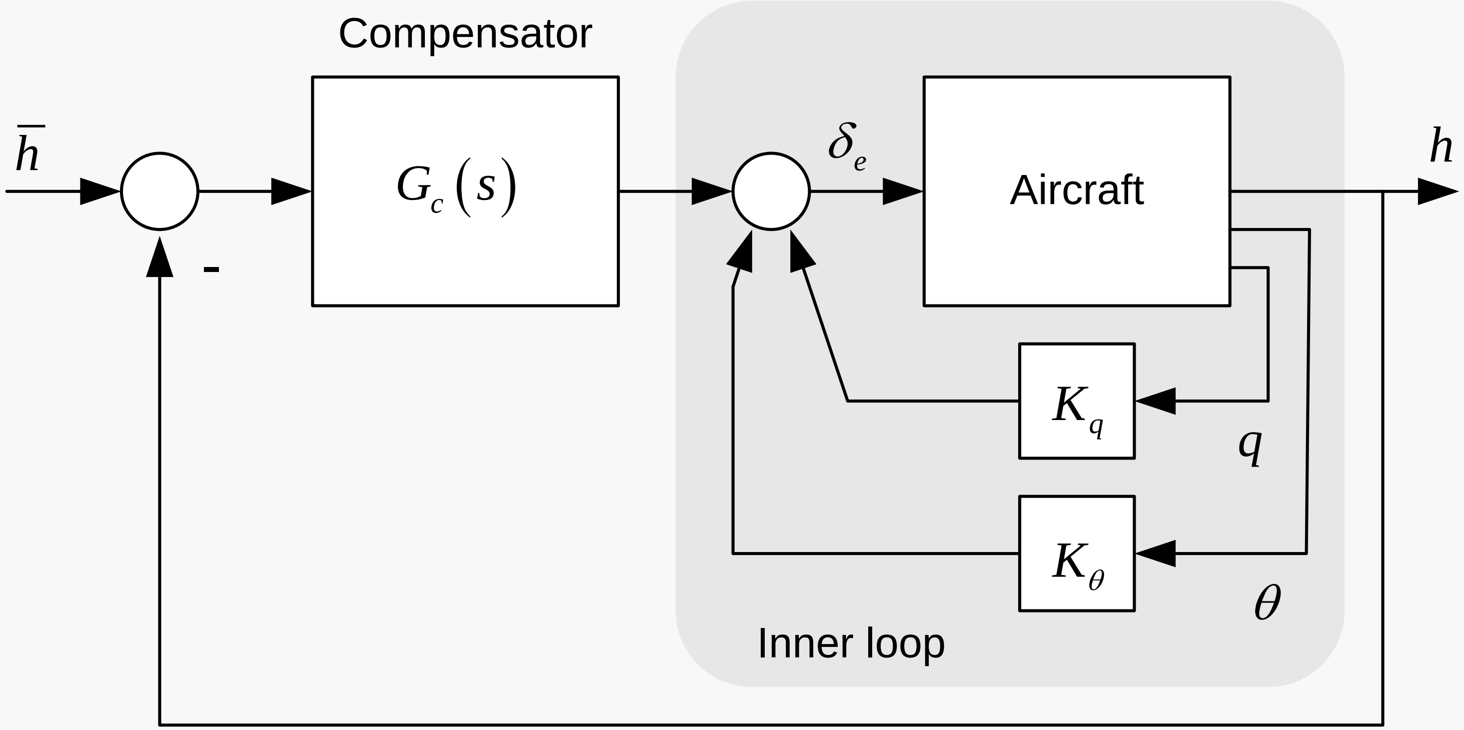
Margins



Bode Diagram
Gm = Inf , Pm = -89 deg (at 0.408 rad/sec)
Magnitude (dB)
20
0
-20
-40
-60
-80
Phase (deg)
135
90
45
0
-45
-90
10^-3
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/sec)

Boeing 747-400
altitude hold controller



Boeing 747 Dynamics (cruise)

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} -0.006 & 0.0263 & 0 & -32.2 & 0 \\ -0.0941 & -0.624 & 820 & 0 & 0 \\ -0.000222 & -0.00153 & -0.668 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 830 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \\ h \end{bmatrix} + \begin{bmatrix} 0 \\ -32.7 \\ -2.08 \\ 0 \\ 0 \end{bmatrix} \delta_e$$

$$h = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \\ h \end{bmatrix}$$

Boeing 747 Open Loop Longitudinal Modes-1

```
>> A=[-0.0064 0.0263 0 -32.2 0;-0.0941 -0.624 820 0 0;
        -.000222 -0.00153 -0.668 0 0;0 0 1 0 0;0 -1 0 830 0];
>> eig(A)
ans =
   0.0000 + 0.0000i
  -0.6463 + 1.1211i
  -0.6463 - 1.1211i
  -0.0029 + 0.0098i
  -0.0029 - 0.0098i
```

- Vertical translation
- Short period
- Phugoid

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Boeing 747 Open Loop Longitudinal Modes-2

```
>> [V,D]=eig(A)

V =
   0.0000   -0.0116 + 0.0037i  -0.0116 - 0.0037i   -0.0358 - 0.0176i  -0.0358 + 0.0176i
   0.0000   -0.9368 + 0.0000i  -0.9368 + 0.0000i    0.0053 + 0.0026i   0.0053 - 0.0026i
   0.0000    0.0000 - 0.0013i   0.0000 + 0.0013i   -0.0000 - 0.0000i  -0.0000 + 0.0000i
   0.0000   -0.0009 + 0.0005i  -0.0009 - 0.0005i    0.0000 + 0.0000i   0.0000 - 0.0000i
   1.0000    0.1816 - 0.2988i   0.1816 + 0.2988i    0.9992 + 0.0000i   0.9992 + 0.0000i
```

Short period

Vertical translation

Phugoid

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Boeing 747 Inner Loop Design

```
A=[-0.0064 0.0263 0 -32.2 0;-0.0941 -0.624 820 0 0;-
.000222 -0.00153 -0.668 0 0;0 0 1 0 0;0 -1 0 830 0];
B=[0;-32.7;-2.08;0;0];
C=[0 0 0 0 1];
poles=[0,-2.25+2.99i,-2.25-2.99i,-0.0105,-0.0531];
Kinner=place(A,B,poles)
Kinner =
   -0.0008   -0.0054   -1.4845   -0.6517         0
eig(A-B*Kinner)
ans =
        0
  -2.2500 + 2.9900i
  -2.2500 - 2.9900i
  -0.0531
  -0.0105
```

K_q K_θ

Small contribution, so we'll drop these two terms. Thus, the implementation does not need an observer.

```
[0,-0.6463+1.1211i,-0.6463-1.1211i,-0.0029+0.0098i,-0.0029-0.0098i]
```

original poles

Boeing 747 cont'd

RHP zero

$$\delta_e \to h : G(s) = \frac{32.7(s+0.0045)(s+0.5645)(s-5.61)}{s(s+0.003\pm j0.0098)(s+0.6463\pm j1.1211)}$$

phugoid short-period

Choose: $K_q = -1.4845,\ K_\theta = -0.6517$

Inner loop improves stability

$$A \to A_p = A + b\begin{bmatrix}0 & 0 & -1.4845 & -0.6517 & 0\end{bmatrix}$$

$$= \begin{bmatrix} -0.0064 & 0.0263 & 0 & -32.2 & 0 \\ -0.0941 & -0.624 & 721 & -21 & 0 \\ -0.0002 & -0.0015 & -3.76 & -1.36 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 830 & 0 \end{bmatrix}$$

New *A* matrix

Note zeros are unchanged

$$G \to G_p(s) = \frac{32.7(s+0.0045)(s+0.5645)(s-5.61)}{s(s+2.25\pm j2.99)(s+0.0105)(s+0.0531)}$$

Outer Loop Design Computations-1

```
>> A=[-0.0064 0.0263 0 -32.2 0;-0.0941 -0.624 761 -196.2 0;
                 -.0002 -0.0015 -4.41 -12.48 0;0 0 1 0 0;0 -1 0 830 0]
A =
   -0.0064    0.0263         0  -32.2000         0
   -0.0941   -0.6240  761.0000 -196.2000         0
   -0.0002   -0.0015   -4.4100  -12.4800         0
         0         0    1.0000         0         0
         0   -1.0000         0  830.0000         0
>> b=[0;-32.7;-2.08;0;0]
b =
         0
  -32.7000
   -2.0800
         0
         0
>> p=[-.0045;-.145;-.513;-2.25+i*2.98;-2.25-i*2.98];
```

Computations 2

```
>> K=place(A,b,p)
K =
   -0.0011    0.0016   -0.0843   -1.6011   -0.0010
>> eig(A-b*K)
ans =
  -2.2500 + 2.9800i
  -2.2500 - 2.9800i
  -0.0045
  -0.5130
  -0.1450
>> c=[0,0,0,0,1];
>> poles=[-0.0045,-5.645,-9,-10,-11];
>> L=place(A',c',poles)'
L =
   -6.5323
  915.2339
   -2.7283
    1.4615
   30.6091
```

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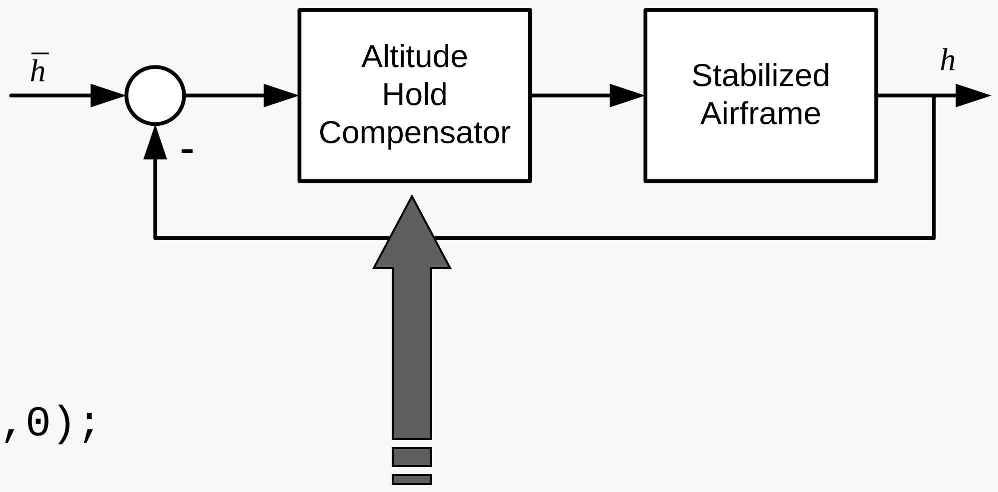
Computations 3

```
>> eig(A-L*c)
ans =
   -0.0045
  -11.0000
  -10.0000
   -9.0000
   -5.6450
>> Ac=A-b*K-L*c;
>> Bc=L;
>> Cc=K;
>> Gcss=ss(Ac,Bc,Cc,0);
>> Gc=tf(Gcss);
>> zpk(Gc)
Zero/pole/gain:
-0.64364 (s+0.5803) (s+0.004708) (s^2  + 4.517s + 14.29)
--------------------------------------------------------
 (s+13.22) (s+5.644) (s+0.004507) (s^2  + 16.9s + 79.2)
```

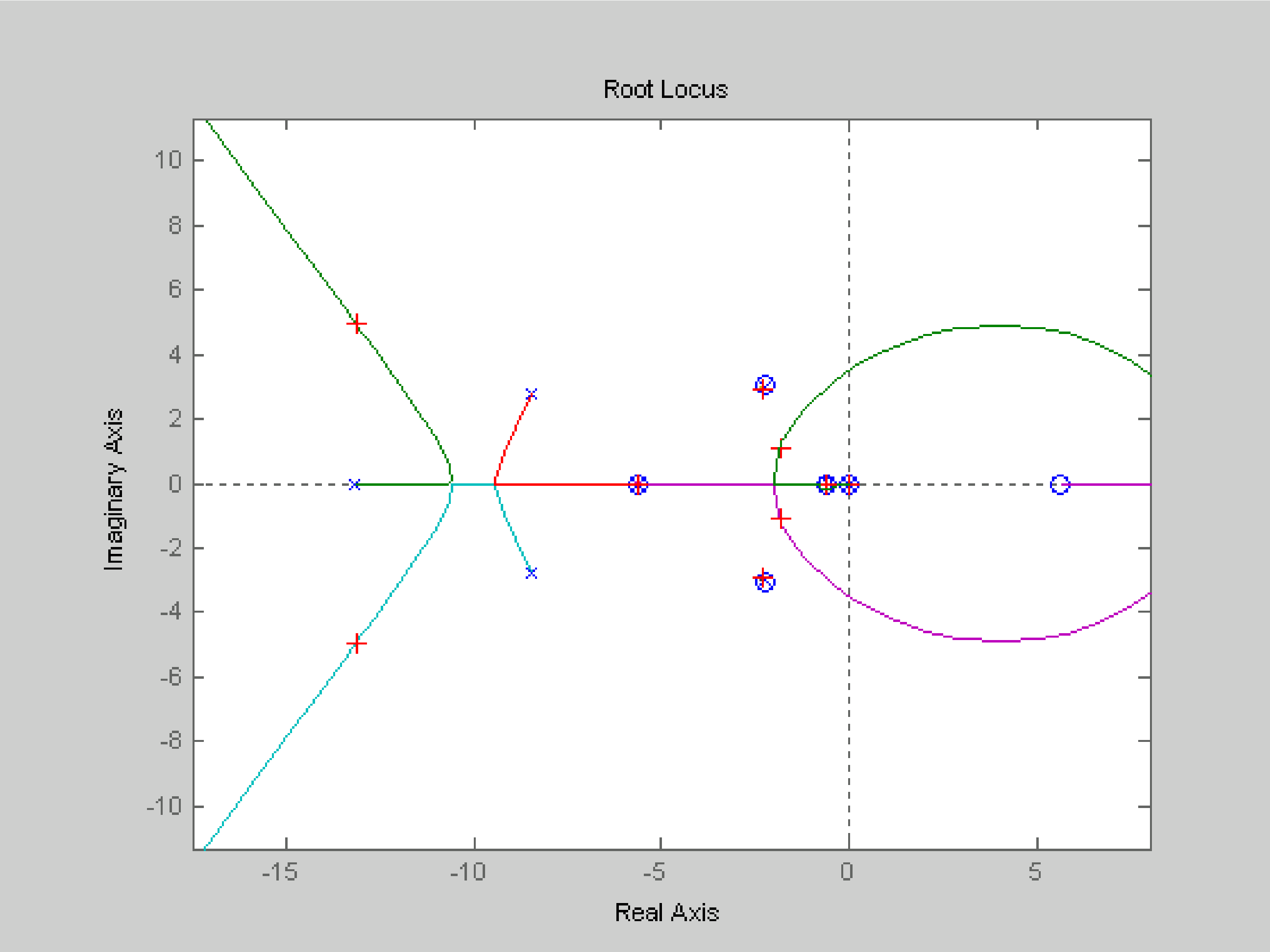


Computations Summary

$$G_p(s) = \frac{32.7(s + 0.0045)(s + 0.5645)(s - 5.61)}{s(s + 2.25 \pm j2.99)(s + 0.0105)(s + 0.0531)}$$

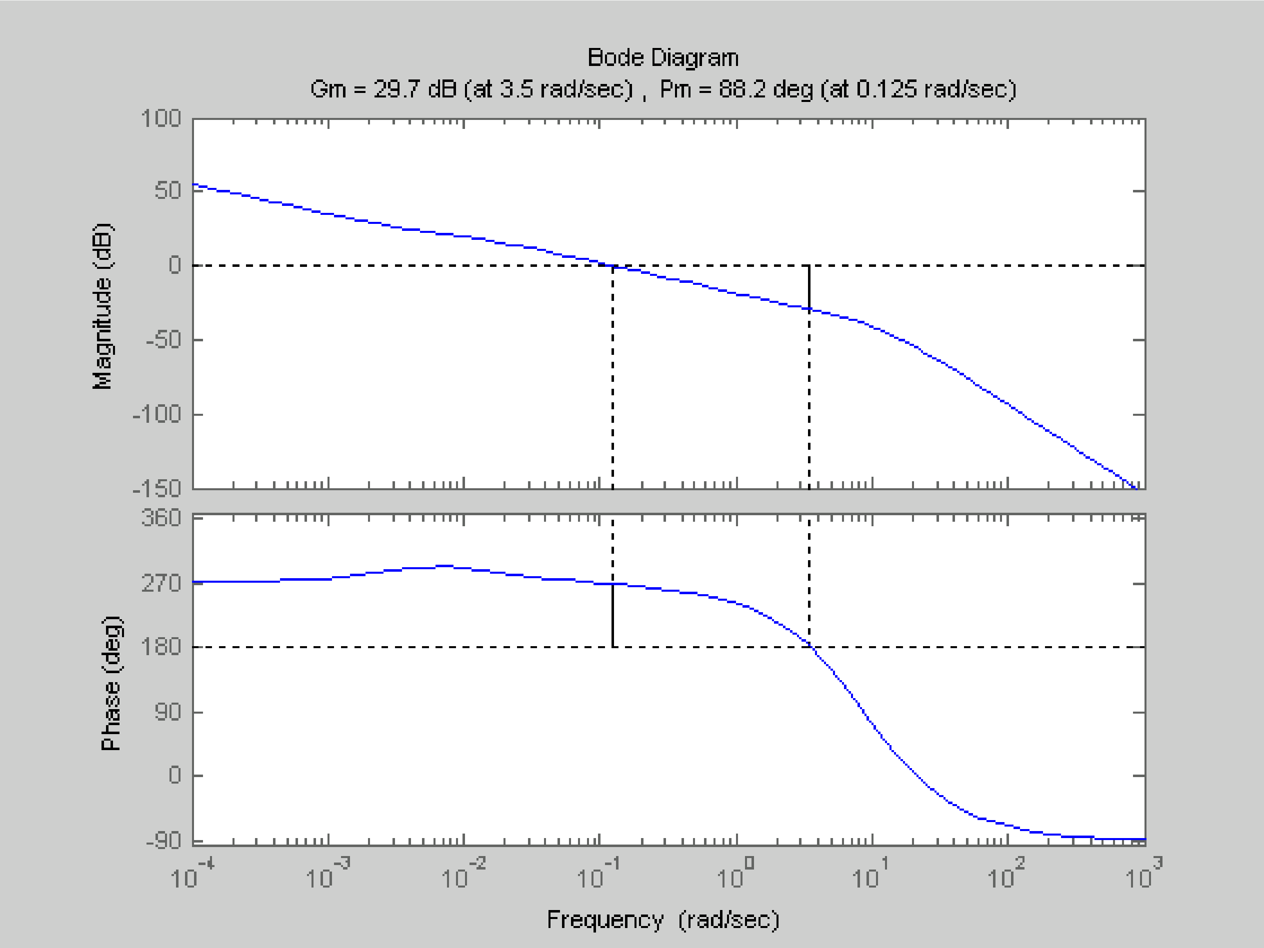
$$G_c(s) = \frac{-0.644(s + 0.5803)(s + 0.004708)(s + 2.258 \pm j3.0314)}{(s + 13.22)(s + 5.644)(s + 0.0045)(s + 8.45 \pm j2.7924)}$$

Root Locus



Root Locus
10
8
6
4
2
0
-2
-4
-6
-8
-10
Imaginary Axis
-15
-10
-5
0
5
Real Axis

Margins



Bode Diagram
Gm = 29.7 dB (at 3.5 rad/sec) , Pm = 88.2 deg (at 0.125 rad/sec)
Magnitude (dB)
100
50
0
-50
-100
-150
Phase (deg)
360
270
180
90
0
-90
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)